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# On Variational Approaches to Steady State Heat Conduction

A. D. SOLOMON

*Tel Aviv University, Department of Mathematics, Tel Aviv, Israel*

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SUMMARY
Several aspects of the variational approach to steady state heat conduction are examined.

## Introduction

Recently Brand and Lahey [1] defined a variational formulation of a steady state heat conduction problem with mixed boundary conditions. In this paper we examine the relationship between some variational and differential problems. In section 1 we remark on this relationship and its principal limitation, and we extend some earlier similar results of Solomon [5] to other steady state heat conduction problems. In section 2 we discuss a method for simplifying the numerical solution of a variational problem. In section 3 we present a simple problem arising in gas dynamics for which the variational approach is of special value.

## 1. The Variational Approach to Steady State Heat Conduction Problems

The variational approach to steady state heat conduction problems is based upon their relationship with and possible equivalence to minimum problems for appropriate integral functionals. The simplest case is that of the Laplace equation

$$T_{xx}+T_{yy}+T_{zz}=0 \tag{1}$$

over a domain $D$ in $(x, y, z)$ space with boundary $\Gamma$, subject to the boundary conditions

$$T\equiv g \quad \text{on} \quad \Gamma\,, \tag{2}$$

for a given function $g$. This corresponds to steady heat conduction in a homogeneous body with constant conductivity and given temperature distribution on the boundary.

It is well known that if a function $G=G(x, y, z)$ exists in $D$ satisfying (2), for which the integral

$$\iiint_D (G_x^2+G_y^2+G_z^2)\,dx\,dy\,dz \tag{3}$$

is finite, then the problem of solving (1) subject to (2) is equivalent to that of minimizing the integral functional

$$I[T]=\iiint_D (T_x^2+T_y^2+T_z^2)\,dx\,dy\,dz$$

among all functions $T(x, y, z)$ satisfying (2). This equivalence may be of great value in computing the solution of (1) subject to (2) and indeed problems exist where it may be nearly essential, as will be seen in section 3. It is important that the engineer having this purpose in mind note that the equivalence does not always hold. In fact simple continuous functions $g$ can be found (see Courant [2]) such that the integral $I[T]$ will be infinite for any function $T$ satisfying (2). Hence the minimum problem for $I[T]$ will have no meaning while a solution to (1), (2) still exists. This fact must be taken into account when wishing to solve steady heat conduction problems via variational formulations.

We will now consider several steady heat conduction problems for composite media, for phase change processes, for bodies with internal heat sources, and for mixed boundary conditions. All of the problems are considered in a domain $D$ of $(x, y, z)$ space with bounding surface $\Gamma$. In each case we define a family $\Omega$ of functions defined in $D$, and an integral functional $I[T]$ for $T=T(x, y, z)$ in $\Omega$ and we consider the problem of minimizing $I[T]$ in $\Omega$:

$$\min_{\Omega}. I[T]. \tag{4}$$

We assume that at least one function $T$ in $\Omega$ exists for which

$$I[T] < \infty. \tag{5}$$

In the first case considered (Problem I) equivalence between the differential and variational problems holds if the conductivity varies only slightly with its argument. For problems II, III the extremal problem (4) and the original problem are equivalent and the equivalence carries over to the non-isotropic case. For each problem we state the differential equation (D.E.) and boundary conditions (B.C.), followed by the integral functional (I.F.) and the class of functions $\Omega$.

Problem I. *A Phase Change Problem*

The domain $D$ is occupied by a material undergoing a change of phase, from phase I to II, at temperature $T_c$. There is an internal heat source $F(x, y, z)$ present; the conductivity $K=K(T)$ is a function only of the temperature and undergoes a jump discontinuity with the phase change; $S$ denotes the interphase surface, while $\Gamma_1$, $\Gamma_2$ denote distinct portions of $\Gamma$, $\Gamma=\Gamma_1+\Gamma_2$. (See Solomon [5])

D.E. $\quad \text{div.}(K \text{ grad. } T)+F=0 \text{ in } D;$ (6)

B.C. $\quad T\equiv g \text{ on } \Gamma_1,$ (7)

$\quad -K \text{ grad. } T\cdot \boldsymbol{n}=Q+aT \text{ on } \Gamma_2;$

$\quad T\equiv T_c \text{ on } S,$ (8)

$\quad K(T)|\text{grad. } T| \text{ is continuous across } S.$

Here $\boldsymbol{n}$ refers to the unit outer normal vector at the corresponding surface; $Q=Q(x, y, z)$ is a known function on $\Gamma_2$; $S$ is the unknown boundary between phase I and phase II domains, whose determination constitutes the main objective of the problem, while $a>0$ is a constant.

The related variational problem is to minimize the integral functional

$$\text{I.F.} \quad I[T]=\iiint_D \{K(T)^2(T_x^2+T_y^2+T_z^2)-2TF\}\,dx\,dy\,dz + \iint_{\Gamma_2}(2QT+aT^2)\,ds \tag{9}$$

for $ds$ the areal element on $\Gamma_2$, over the class $\Omega$ of all functions $T$ such that

(1) $I[T]<\infty$

(2) $T=g$ on $\Gamma_1$.

The interface conditions (8) automatically determine the surface $S$.

Problem II. *A Composite Medium*

The domain $D$ is the union of two subdomains $D_1$ and $D_2$, occupied by different materials. The conductivity $K=K(x, y, z)$ is a function of $(x, y, z)$ alone, and undergoes a jump discontinuity across the surface $S$ between the two domains. We assume that a heat source given by the function $F(x, y, z)$ is again present.

D.E. $\quad \text{div.}(K(x, y, z) \text{ grad. } T)+F=0 \text{ in } D_1, D_2;$ (10)

B.C. $\quad T\equiv g \text{ on } \Gamma_1;$ (11)

$$-K \text{ grad. } T\cdot \boldsymbol{n} = Q + aT \quad \text{on } \Gamma_2 . \tag{12}$$

$K$ grad. $T\cdot \boldsymbol{n}$ is continuous across $S$.

An equivalent minimum problem refers to the functional

$$\text{I.F.} \qquad I[T] = \iiint_D \{K(T_x^2+T_y^2+T_z^2)-2TF\}\,dx\,dy\,dz + \iint_{\Gamma_2} (2QT+aT^2)\,ds \tag{13}$$

over the class of all functions $\Omega$ such that

(1) $I[T]<\infty$,
(2) $T\equiv g$ on $\Gamma_1$.

As in problem I the condition (12) is automatically satisfied by the solution to the minimum problem.

Problem III. *A Heat Source Distributed Along an Internal Surface*

For the sake of simplicity we consider a homogeneous medium with unit conductivity. Let $S$ be a surface lying entirely within $D$. We wish to find the steady temperature distribution arising under the following conditions:

D.E. $\quad T_{xx}+T_{yy}+T_{zz}=0$ in $D$,

B.C. $\quad T\equiv g$ on $\Gamma_1$,

grad. $T\cdot \boldsymbol{n} = Q$ on $\Gamma_2$,

$[\text{grad. } T\cdot \boldsymbol{n}]_S = -R$ on $S$,

where $[\ \ ]_S$ denotes the limiting value of the enclosed function at $S$ from the side of $S$ into which $\boldsymbol{n}$ points, minus that from the opposite side; $Q=Q(x,y,z)$ and $R=R(x,y,z)$ are given functions.

The equivalent variational problem is to minimize the integral functional

$$\text{I.F.} \qquad I[T]=\iiint_D (T_x^2+T_y^2+T_z^2)\,dx\,dy\,dz + 2\iint_{\Gamma_2} QT\,ds + \iint_S RT^2\,ds$$

in the class $\Omega$ of all functions $T$ such that $T\equiv g$ on $\Gamma_1$. The conditions at $S$ hold automatically for the solution.

## 2. A Method of R. Courant Simplifying the Numerical Solution of the Variational Problem

One of the advantages of variational formulations of steady state heat conduction problems is the availability of additional methods for their solution, such as that of Rayleigh–Ritz. A difficult problem in applying these methods is the satisfaction of the condition

$$u\equiv g \quad \text{on } \Gamma_1 . \tag{14}$$

One method of avoiding this difficulty is the addition to the functional $I[u]$ of a term of the form

$$\lambda \iint_{\Gamma_1} (T-g)^2\,ds \tag{15}$$

for $\lambda>0$ a large positive constant, and the consideration of the minimum problem for this new augmented functional in a class $\Omega$ of functions for which we no longer require the condition

(14). If $T^{(\lambda)}$ denotes the solution to this altered problem one can show that $T^{(\lambda)}$ converges to the solution of our original problem, satisfying (14), as $\lambda \to \infty$. This will be shown for a simpler problem, without loss of generality, in the Appendix. The method of introducing (15) has been seen to furnish good results by R. Courant [3].

## 3. A Problem of Elliptic-Parabolic Type

In recent years problems of transonic gas dynamics and magnetohydrodynamics have been of increasing interest. These fields yield differential equations which may degenerate, for example from elliptic to parabolic type, at the boundary of a domain. In such problems the engineer cannot simply utilize standard finite difference methods, since the data, although sufficient for defining the solution, is insufficient for using these methods. (See i.e. Jamet [4]). In these cases variational methods are of special value. We will consider the following example: (See also A. Solomon and F. Solomon [6])
*Problem*: Solve the equation

$$\text{D.E.} \quad (tu_t)_t + u_{xx} = 0 \text{ in } D : 0 \leqq x, t \leqq 1 \tag{16}$$

for the function $u(x, t)$ satisfying the conditions

$$\begin{aligned} \text{B.C.} \quad & u(0, t) = \phi_1(t), && 0 \leqq t \leqq 1 \\ & u(x, 1) = f(x), && 0 \leqq x \leqq 1 \\ & u(1, t) = \phi_2(t), && 0 \leqq t \leqq 1 . \end{aligned} \tag{17}$$

Note that no value is specified for $u(x, 0)$, but on the contrary, this value can be shown to be uniquely determined by (16), (17). Conditions (16) and (17) are clearly insufficient for the application of any of the usual finite difference methods for solving elliptic equations. Nevertheless it is easily seen that the problem (16), (17) is equivalent to the minimum problem for the functional

$$\text{I.F.} \quad I[u] = \iiint_D (tu_t^2 + u_x^2)\, dt\, dx$$

in the class of all functions satisfying (17), with the function value $u(x, 0)$ not specified.

## Appendix

Consider the problem

$$\min_{\Omega} .\ I[T] \tag{18}$$

for

$$I[T] = \iiint_D (T_x^2 + T_y^2 + T_z^2)\, dx\, dy\, dz + \lambda \iint_\Gamma (T - g)^2\, ds$$

among all functions $T$ with no limitation upon the behavior of $T$ on $\Gamma$, where $g$ is defined on $\Gamma$. If $T = T^{(\lambda)}$ is the solution to the problem (18), then

$$\frac{d}{d\varepsilon} I[T + \varepsilon\eta]\bigg|_{\varepsilon=0} = 0$$

for any function $\eta$. This implies

$$\iiint_D (T_x\eta_x + T_y\eta_y + T_z\eta_z)\, dx\, dy\, dz + \lambda \iint_\Gamma \eta(T - g)\, ds = 0 .$$

Assume that $G$ is a function defined on $D$, and coincident with $g$ on $\Gamma$. Let $\eta = T - G$. Then

$$\lambda \iint_{\Gamma} (T-g)^2 ds = -\iiint_D (T_x^2+T_y^2+T_z^2)\,dx\,dy\,dz + \iiint_D (T_x G_x + T_y G_y + T_z G_z)\,dx\,dy\,dz\,.$$

For any number $\varepsilon > 0$, with $\varepsilon \leqq \sqrt{2}$

$$T_x G_x \leqq \tfrac{1}{2}\varepsilon^2 T_x^2 + G_x^2/2\varepsilon^2$$

with similar inequalities for $T_y G_y$ and $T_z G_z$. Hence

$$\iint_{\Gamma} (T-g)^2 ds \leqq (1/2\lambda\varepsilon^2) \iiint_D (G_x^2+G_y^2+G_z^2)\,dx\,dy\,dz = 0(1/\lambda)\,.$$

As $\lambda \to \infty$, $T = T^{(\lambda)} \to g$ on $\Gamma$ and hence $T^{(\lambda)}$ converges to the solution of the equation

$$T_{xx} + T_{yy} + T_{zz} = 0 \text{ on } D\,,$$

satisfying (14) on $\Gamma$.

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